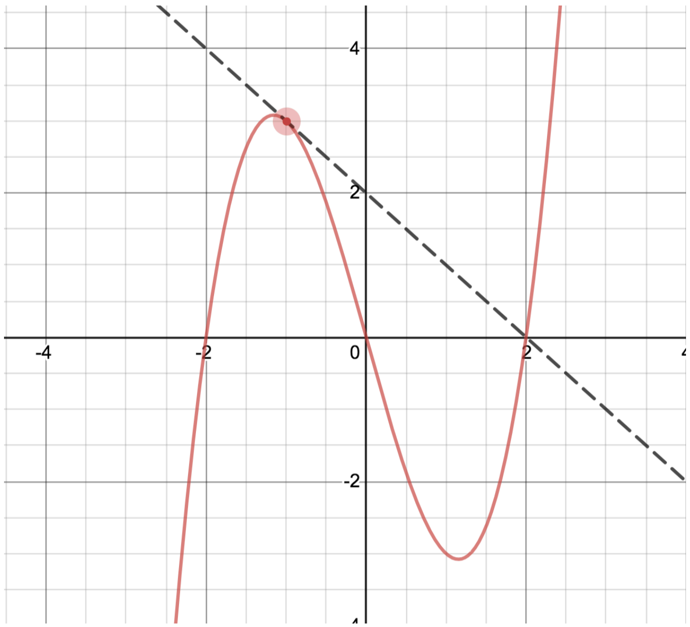
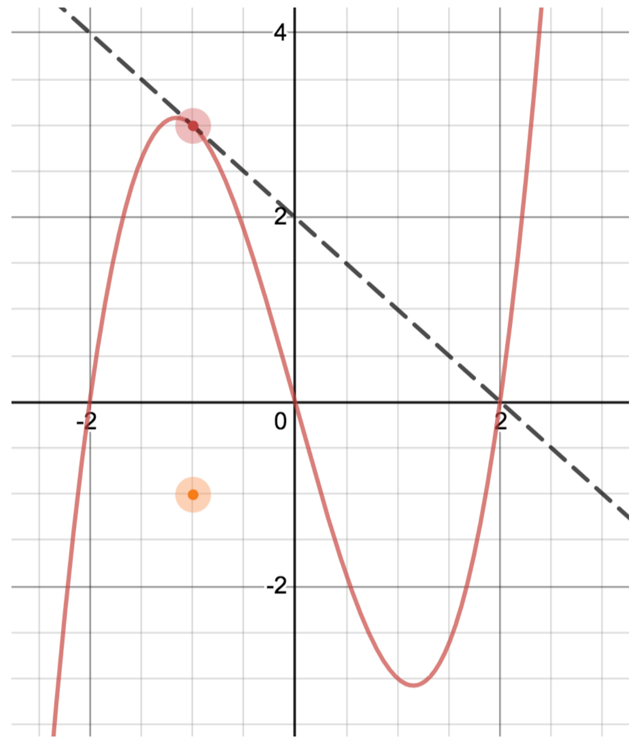
2.2 The Derivative as a Function

The following example is based on the Desmos Animation from the 5A page.

Use the graph of f(x) below to estimate $f'(-1)$





x	
-1	
-2	
0	
1	
2	

Now draw a point on the graph that as x value of -1 and y value that is $f'(-1)$.

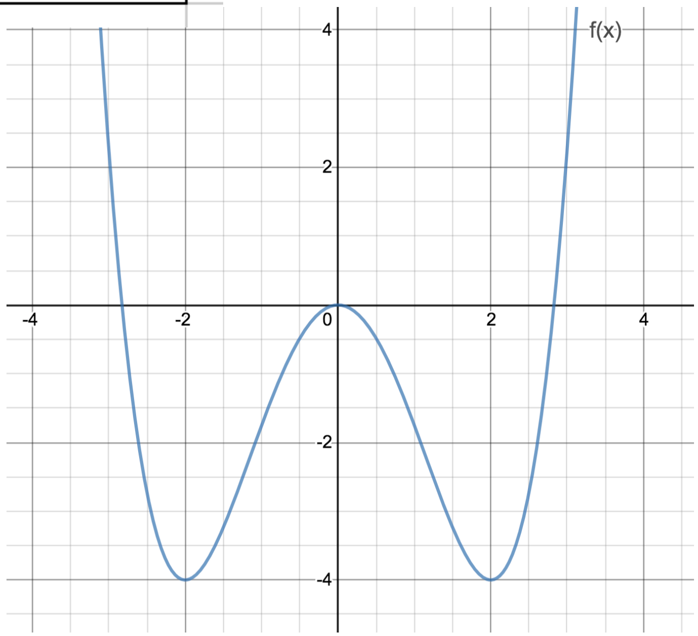
Repeat the above steps for x=-2,-0,1,2

We have created a new function $f'(x)$

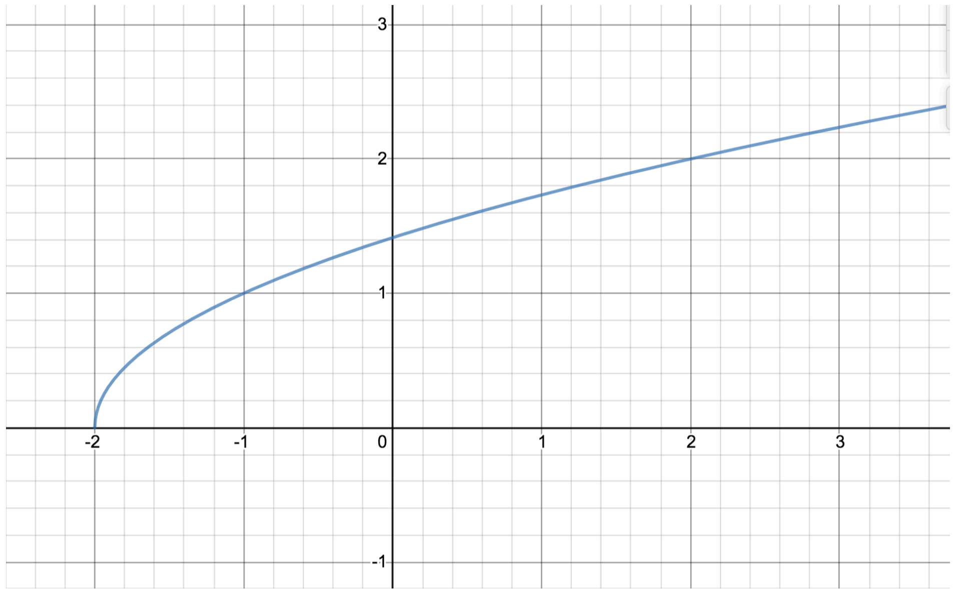
The ______________ value on the graph of $f'(x)$ corresponds to the __________________ of the graph of $f(x)$.

When	The graph of f(x) has	The graph of f'(x) has
f'(x)=0		
f'(x)>0		
f'(x)<0		

Given the graph of $f(x)$, sketch the graph of $f'(x)$.



Example (1) : Given the graph of $f(x) = \sqrt{x+2}$,sketch the graph of $f'(x)$.



Now compute $f'(a)$. Look at the estimated graph vs. the algebraic computation.

Substituting x for a in $f'(a)$ we get $f'(x) =$

What is the domain of $f'(x)$? _______________ What happens at x=-2?____________

Generalizing definitions:

Derivative at a specific point:	General derivative function:
$f'(a) == \lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$	$f'(x) == \lim_{h\to 0}$
$f'(a) = \lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}$	$f'(x) = \lim_{z\to x}$

PROVIDED THE LIMIT EXISTS

If the limit does not exist for x=a, we say $f(x)$ is not differentiable at x=a.

In the above example, $f(x)$ fails to be ________________________ at x=-2. $f(x)$ IS differentiable on the interval ______________________

Alternate notation for $f'(x)$

If $y = f(x) = \sqrt{x+2}$ then we write $f'(x) = \dfrac{dy}{dx} = \dfrac{df}{dx} = \dfrac{d}{dx}\left[\sqrt{x+2}\right] = \dfrac{1}{2\sqrt{x+2}}$, or if $w = s(t) = \sqrt{t+2}$, we would write $s'(t) = \dfrac{dw}{dt} = \dfrac{ds}{dt} = \dfrac{d}{dt}\left[\sqrt{t+2}\right] = \dfrac{1}{2\sqrt{t+2}}$

Think of $\dfrac{dy}{dx}$ as a ____________________ and $\dfrac{d}{dx}$ as a ____________________________

So we would not write $\dfrac{dy}{dx}\left[\sqrt{x+2}\right]$ nor $\dfrac{d}{dx} = \dfrac{1}{2\sqrt{x+2}}$

Example 2: $f(x) = |x|$, find $f'(x)$ (or find ________________)

Example 3: $f(x) = \begin{cases} x^2 & if \quad x \geq 1 \\ 3 & if \quad x < 1 \end{cases}$, find $f'(x)$

Examples 1, 2, and 3 show the three ways in which a function fails to be differentiable at a given point:

1)______________________________

2)______________________________

3)______________________________

Algebraically, to determine where f is differentiable, we find the domain of $f'(x)$

Theorem: IF $f(x)$ is differentiable at x=a, THEN $f(x)$ is continuous at x=a

(Differentiability at x=a $\Longrightarrow$ Continuity at x=a)

<u>Logic Basics:</u>

Conditional Statement:

Converse:

Contrapositive:

<u>Proof:</u>

Given that $f(x)$ is differentiable at x=a, we know $\lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a} = f'(a)$

:

:

:

Higher derivatives

Since $f'(x)$ is a function itself, we can take the derivative of it to create a new function which we denote ____________________

Using Leibniz notation, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ = __________________________

Likewise, we can take higher order derivatives: __________________________

Application: If s(t) is the position function of an object that moves in a straight line, then

$$s'(t) = \frac{ds}{dt} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$s''(t) = \frac{d^2s}{dt^2} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

2.3i Differentiation Formulas

We can easily show

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x) = 1 \qquad \frac{d}{dx}\left(x^2\right) = 2x \qquad \frac{d}{dx}\left(x^3\right) = 3x^2$$

Theorem: If n is a positive integer then $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

Proof: (see book for a different approach):

$$\frac{d}{dx}\left(x^n\right) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \underline{\hspace{8em}} =$$

We will learn later that this rule applies for all real exponents n.

<u>Examples</u>:

$$\frac{d}{dx}\left(x^5\right) = \underline{\hspace{6em}} \qquad \frac{d}{dt}\left(x^7\right) = \underline{\hspace{6em}} \qquad \frac{d}{dx}\left(\sqrt{x}\right) = \underline{\hspace{6em}} \qquad \frac{d}{dx}\left(\frac{1}{x^2}\right) = \underline{\hspace{6em}}$$

Important notation:

$\frac{d}{dx}$ is an operator. It is telling us to take the derivative It always has an argument. $\frac{d}{dx}(\textit{something})$

$\frac{dy}{dx} = f'(x) = y'$ is the name of the derivative.

So if $f(x) = x^4$ we can say $f'(x) =$ ______________ or $\frac{dy}{dx} = 4x^3$ or $y' = 4x^3$ or $\frac{d}{dx}\left(x^4\right) =$ __________

But it is INCORRECT to say $\frac{d}{dx} = 4x^3$ or $\frac{dy}{dx}\left(x^4\right) = 4x^3$

Also, arguments should match: $f'(x) = 4x^3$, $f'(a) = 4a^3$

The Constant Multiple Rule: If c is a constant and f(x) is a differentiable function then $\frac{d}{dx}\left(cf(x)\right) = c\frac{d}{dx}\left(f(x)\right)$

Example:

Proof:

The Sum and Difference Rule: If f(x) and g(x) are differentiable then $\frac{d}{dx}\left(f(x) \pm g(x)\right) =$ ____________________

Example:

See proof in book.

The Product Rule: If f(x) and g(x) are differentiable then $\frac{d}{dx}\left(f(x)g(x)\right) =$ ______________________________

Often abbreviated: $(fg)' =$ __________________

Derivation: (differs from book)

$$\frac{d}{dx}\left(f(x)g(x)\right) = \lim_{z\to x}\frac{f(z)g(z)-f(x)g(x)}{z-x} = \lim_{z\to x}\frac{f(z)g(z) \qquad\qquad\qquad\qquad -f(x)g(x)}{z-x} =$$

Example:

The Quotient Rule: If f(x) and g(x) are differentiable then $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) =$ ______________________________

Often abbreviated: $\left(\frac{f}{g}\right)' =$ __________________

Examples: Find $f'(x)$:

$$f(x)=\frac{2x-3}{x^2+1}$$

$$f(x)=\frac{\sqrt{x}-1}{\sqrt{x}+1}$$

Tip: Many functions are easier to differentiate if you simplify first and write as a power function when possible rather than immediately apply the product or quotient rule.

Examples: Find $f'(x)$:

$$f(x)=\frac{5x}{4}$$

$$f(x)=\frac{3}{x^2}$$

$$f(x)=\sqrt[3]{x}\left(8x^3+x\right)$$

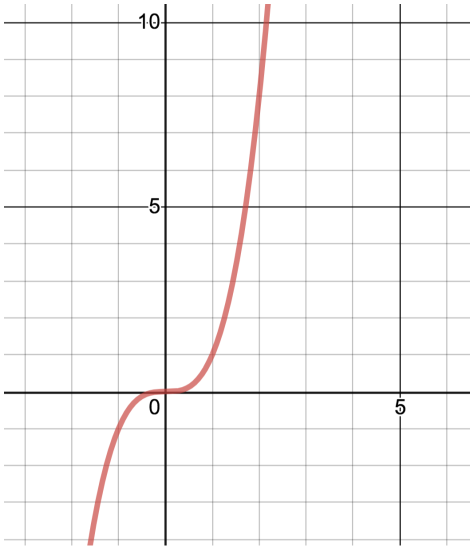
$$f(x)=\frac{4x^3-7x}{\sqrt{x}}$$

2.3ii More challenging tangent problems

Example:

1) Find the equation of the tangent line to $f(x) = x^3$ at x=2

2) Find the equation of the tangent line to $f(x) = x^3$ that contains the point (2,0)



See Example 2.3ii on 5A page:
https://www.desmos.com/calculator/nc9gc2jdqq

2.4i Derivation of Derivatives of the Trigonometric Functions and special limits

Find a formula for the derivative of $f(x)=\cos(x)$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\qquad\qquad\qquad}{h}$$

$$=\lim_{h\to 0}\frac{\qquad\qquad\qquad}{h}$$

$=$

(see below)

$=$

$=$

(see next page)

$=$

Similarly, we can show $\frac{d}{dx}[\sin x] =$ ______________

With these formulas, all the other trigonometric derivatives can be easily found:

Find $\frac{d}{dx}[\tan x] =$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

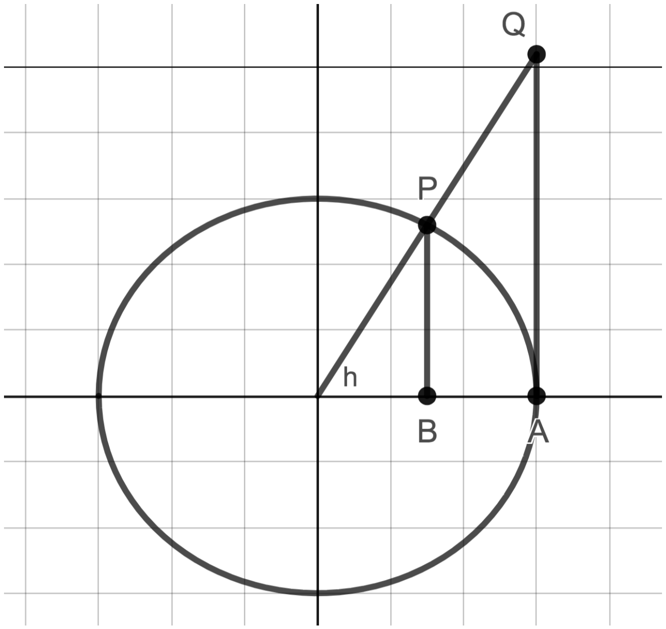
$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Examples: Find $\frac{dy}{dt}$ if $y = t^3 \tan t$ Find $\frac{d}{dx}\left(\frac{5x^2}{\sec x}\right)$ Find $g'(y)$ if $g(t) = y \sin y \cos y$

2.4ii Limits

Finding the limit $\lim_{h\to 0}\frac{\sin(h)}{h}$ and $\lim_{h\to 0}\frac{\cos(h)-1}{h}$
(Development differs from book)



The from here, $\lim_{h\to 0}\frac{\cos(h)-1}{h} =$

Examples: $\lim_{x\to 0}\frac{\sin(3x)}{x}$ $\lim_{x\to 0}\frac{\tan(x)}{x}$

Note Outlines 2.5 – The Chain Rule

SUPER IMPORTANT TO MASTER THIS SECTION

How would we find the derivative of f(x)=sin(2x)?

Need a way to find the derivative of composite functions

Review of compositions of functions

Ex: If $f(x)=\cos(x)$ and $g(x)=\sqrt{x}$, find $(f \circ g)(x)=f(g(x))$ and $(g \circ f)(x)=g(f(x))$

Going backwards, decompose the following functions:

$h(x)=(2x+3)^2$ $p(x)=\ln(\sin(x))$ $r(x)=\tan^2\left(x^3\right)$

We will need to be able to recognize when we have a composite function and we will need to identify from the outermost function to the innermost function.

Development of the Chain Rule

Let $F(x) = (f \circ g)(x) = f(g(x))$. We need to find the derivative.

$$F'(a) = \lim_{x \to a} \frac{F(x) - F(a)}{x - a} = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \quad .$$

Make a temporary substitution for visual help, let v=g(x) and u=g(a), then

$$F'(a) = \lim_{x \to a} \frac{f(v) - f(u))}{x - a} = \lim_{x \to a} \frac{f(v) - f(u))}{\qquad} \bullet \frac{\qquad}{x - a}$$

<u>Example using the notation precisely:</u>

Given F(x)= sin(2x), find F’(x)

How we do this in practice.

$$\frac{d}{dx} \underbrace{f}_{\text{outer function}} \underbrace{(g(x))}_{\text{evaluated at inner function}} = \underbrace{f'}_{\text{derivative of outer function}} \underbrace{(g(x))}_{\text{evaluated at inner function}} \cdot \underbrace{g'(x)}_{\text{derivative of inner function}}$$

$\frac{d}{dx}[\sin(2x)]$ =

$\underbrace{\qquad}_{\text{derivative of outer function}} \quad \underbrace{\qquad}_{\text{evaluated at inner function}} \qquad \underbrace{\qquad}_{\text{derivative of inner function}}$

Another way of thinking about it: If the inner function WAS x…….

$$\frac{d}{dx}[\sin(2x)]$$

Example: Find $f'(x)$ (2 "layers", apply chain rule once)

(1) $f(x) = \sqrt{x^2 + 1}$

(2) $f(x) = \cos^2 x$

(3) $f(x) = \sec\left(x^3\right)$

Example: (3 "layers", apply chain rule twice) $f(x) = \sqrt{\tan(5x^2)}$

Example: (Using chain rule instead of quotient rule): $f(x)=\frac{3x+5}{7x-1}$

Examples: (Combining all the differentiation rules.)

(1) $f(x)=\frac{x^3}{\sqrt{x^2-4}}$

(2) $f(x)=x^4\cos(6x)$

(3) $f(x) = \cos^2 x + \sin^2 x$

Examples: (Abstract) Find $\frac{dy}{dx}$: $y = f\left(x^2\right)$ $y = \left(f(x)\right)^2$

2.6 – Implicit Differentiation

Review notation

$\frac{d}{da}(a^4)$ = $\frac{d}{dx}(a^4)$ = $\frac{d}{dy}(\sin(y)$ = $\frac{d}{dx}(\sin(y)$ =

Review chain rule on abstract problem:

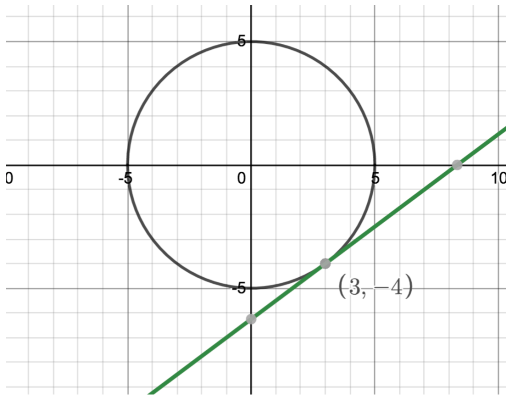
$\frac{d}{dx}[f(x)]^2$ $\frac{d}{dx}[\sin(f(x))]$ $\frac{d}{dx}\left[x^2\sqrt{f(x)}\right]$

Reconsider $\frac{d}{dx}(\sin(y)$ What if y was representing a function of x, $y = f(x)$?

Then $\frac{d}{dx}(\sin(y)$ = $\frac{d}{dx}[\sin(f(x))]$ =

Motivating Problem: Find the slope of the tangent line to $x^2+y^2=25$ at the point (3,-4)

<u>Explicitly</u>



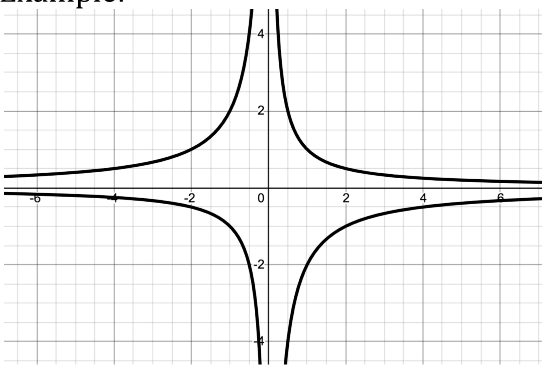
Implicitly: $x^2+y^2= \quad 25$

Example: Find $\frac{dy}{dx}$: $5y^2+\sin y=x^2$

\

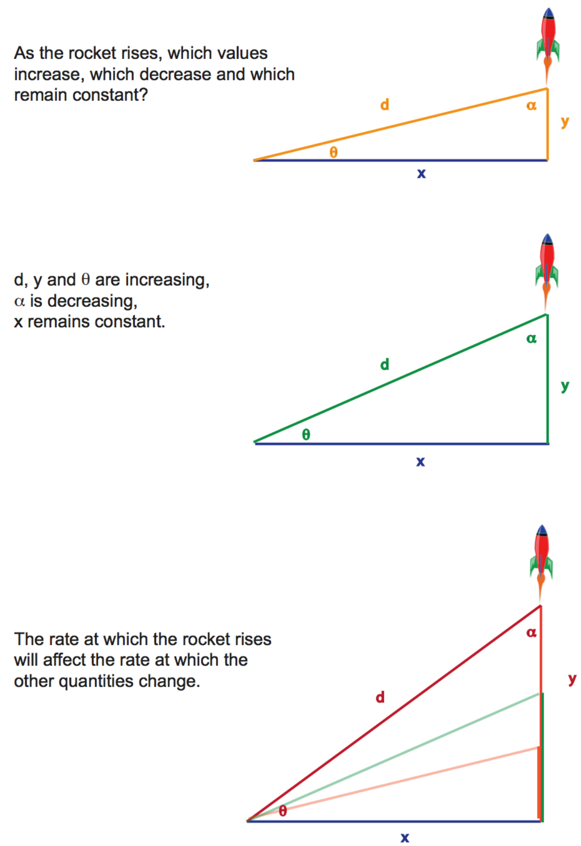
59. Find all points on the curve $x^2y^2 + xy = 2$ where the slope of the tangent line is -1.

Example:



2.8 Related Rates

Consider the following problem. A rocket is rising vertically, and an observer watches at a distance of x feet away.



What do each of the following represent, physically?
Are they positive or negative? If distance is measured
In feet and time in seconds, what are the units?

$$\frac{dy}{dt}, \quad \frac{dD}{dt}, \quad \frac{d\theta}{dt}, \quad \frac{d\alpha}{dt}, \quad \frac{dx}{dt}$$

These rates are all related.

43. A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft.

(a) How fast is the distance from the television camera to the rocket changing at that moment?

(b) If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?

Sketch a picture. Put in any values which are NOT changing. Assign a variable (and label) to any quantities of interest that ARE changing.

(a) What rate do you know? What rate do you WANT to know?

Create an equation relating ________________ and ________________________.

Differentiate both sides with respect to t.

Evaluate at the specified moment of interest. A separate picture for that moment in time might help.

(b)

Example:

14. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

Example: Water is flowing into a conical tank which has height of 16 cm and radius of 4 cm at a rate of 2 cm^3/min. How fast is the water level rising when it is 10 cm deep?

Example 6 A light is on the top of a 12 ft tall pole and a 5ft 6in tall person is walking away from the pole at a rate of 2 ft/sec.

(a) At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole?

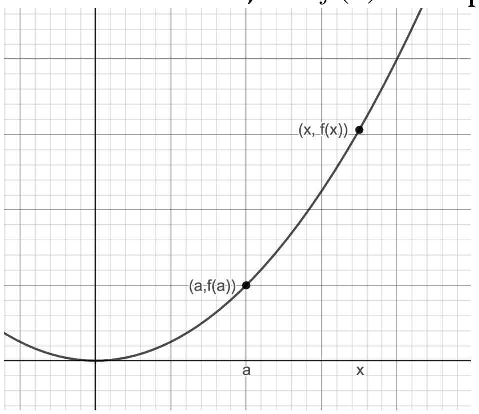
(b) At what rate is the tip of the shadow moving away from the person when the person is 25 ft from the pole?

48. Two people start from the same point. One walks east at 3 mi/h and the other walks northeast at 2 mi/h. How fast is the distance between the people changing after 15 minutes?

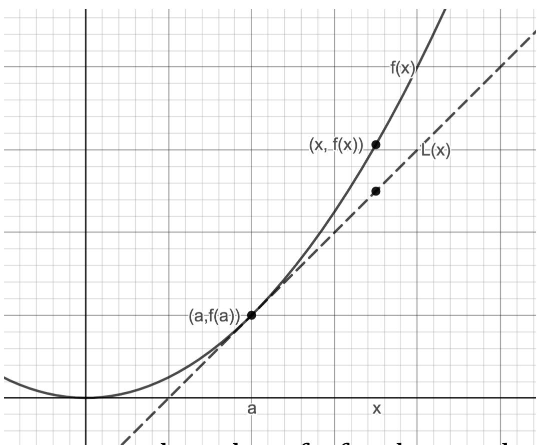
2.9 Linear Approximation and Differentials

Linear Approximation:

Suppose we are given a function and we can easily compute $f(a)$. Further suppose we want to compute the value of $f(x)$ for an x value near a, but $f(x)$ is complicated to compute for that value of x.

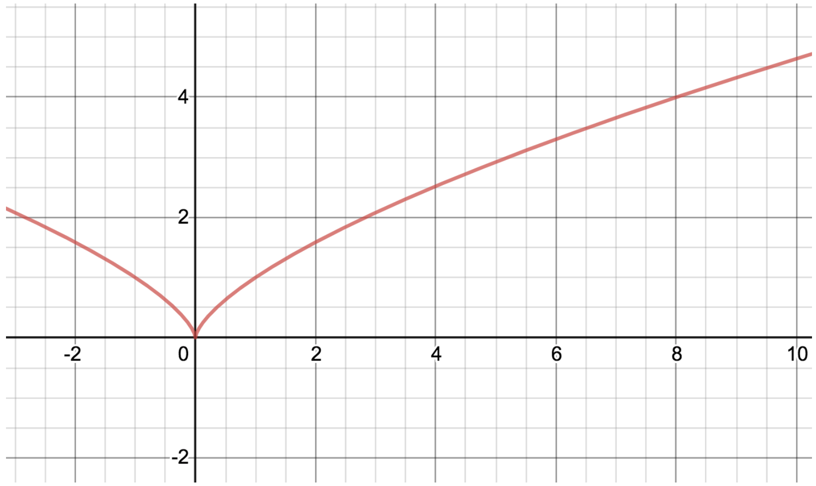


Find the equation of the line tangent to $f(x)$ at x=a



We can compute the value of y for that x value on the tangent line easily, and use it to approximate the y value for that x on f(x). That is $L(x) \approx f(x)$ for x “near a”.

Example: Find a linear approximation for $(7.94)^{2/3}$



Let $f(x) =$ _______________ We want to compute __________________

Choose a. This should be a number near ______________ such that $f(a) = a^{2/3}$ is easy to compute. $a =$______________

Find the equation of the line tangent to $f(x)$ at x=a.

Find the y value of the point on the tangent line corresponding to x=7.94.

(Calculator Value $(7.94)^{2/3} \approx 3.979974916$)

Higher degree polynomial approximations. (See Taylor Series Desmos on 5B page)

Differentials

The derivative, $f'(x)$ can also be written as ________________________

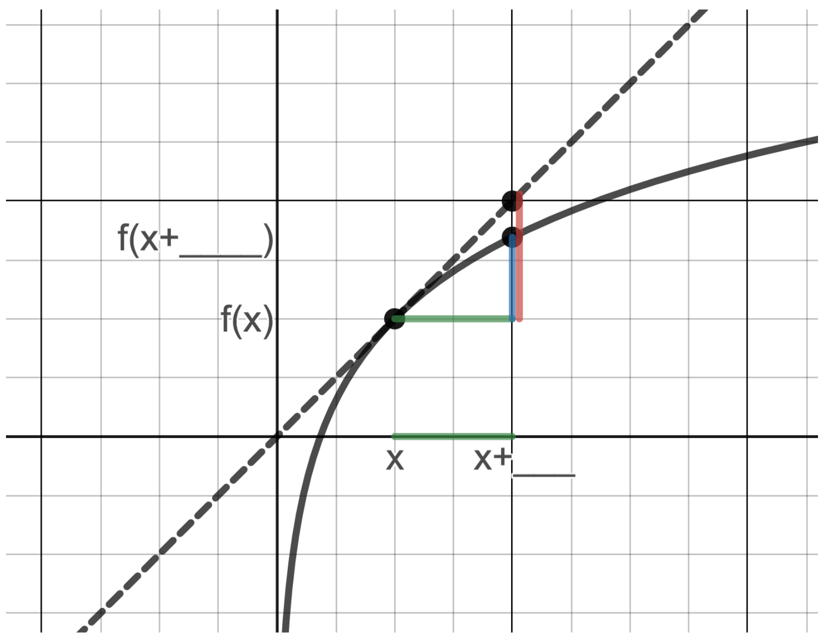
But do dy, dx have any meaning individually?

Definition: The ______________________________, dy, is defined in terms of the differential, dx as:

$$dy = f'(x)dx$$ (dx is treated as an independent variable)

<u>Example</u>: If $y = 2x^3 + \sin x$ then dy=________________________ When x=0 and dx=0.2, dy=_____________________________

Graphical explanation of dy.



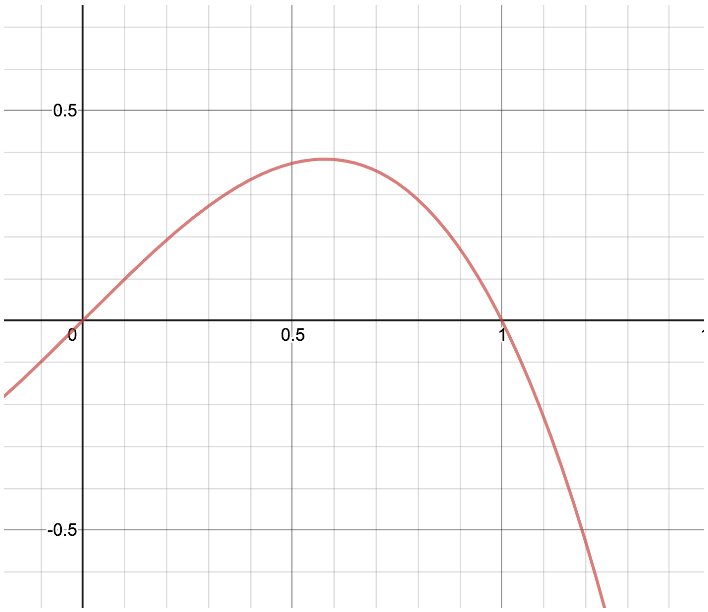
If we let $dx = \Delta x$, then for small values of dx, _________________________________ Differentials are used in two ways;

1)

2)

1) Using differentials to approximate Δy

Example: Given $y = x - x^3$, compute and compare Δy and dy for x=1, $\Delta x = -0.2$ Show Δy and dy on the graph.



Example:

32. The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm.
(a) Use differentials to estimate the maximum error in the calculated area of the disk.
(b) What is the relative error? What is the percentage error?

2) <u>Using differentials to approximate</u> $f(x+\Delta x)$ (linear approximation)

Example: Use differentials to find a linear approximation for $(7.94)^{2/3}$

Let $f(x)=$ ________________ We want to compute ____________________

Choose x. This should be a number near _______________ such that $f(x)=x^{2/3}$ is easy to compute. x=_______________ (x plays the role of "a" when using L(x)). Then

$x+\Delta x=$ ________________________ so $\Delta x=$ _________________

Example: Use differentials to approximate $(7.94)^{2/3}$